

Covariant formulation of Noether's Theorem for translations on κ -Minkowski spacetime

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Abstract

The problem of finding a formulation of Noether's theorem in noncommutative geometry is very important in order to obtain conserved currents and charges for particles in noncommutative spacetimes. In this paper, we formulate *Noether's theorem* for translations of κ -Minkowski noncommutative spacetime on the basis of the *5-dimensional κ -Poincaré covariant* differential calculus. We focus our analysis on the simple case of free scalar theory. We obtain *five* conserved Noether currents, which give rise to *five* energy-momentum charges. By applying our result to plane waves it follows that the energy-momentum charges satisfy a special-relativity dispersion relation with a generalized mass given by the fifth charge. In this paper we provide also a rigorous derivation of the equation of motion from Hamilton's principle in noncommutative spacetime, which is necessary for the Noether analysis.

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1 Introduction

Noncommutative spacetime was introduced in Ref. [1, 2] in order to improve the singularity of quantum field theory at short distances. Afterwards, the idea of noncommutative structure of spacetime inspired several approaches to Quantum Gravity[3]. In particular, Doplicher *et al.* [4, 5] explored the possibility that Quantum Gravity corrections can be described algebraically by replacing the traditional (Minkowski) spacetime coordinates x_μ by Hermitian operators $\hat{\mathbf{x}}_\mu$ ($\mu = 0, 1, 2, 3$) which satisfy the nontrivial commutation relations

$$[\hat{\mathbf{x}}_\mu, \hat{\mathbf{x}}_\nu] = i\theta_{\mu\nu}(\hat{\mathbf{x}}). \quad (1.1)$$

The model of κ -deformed spacetime considered in this paper and denoted by the name of κ -Minkowski spacetime is a particular realization of Eq. (1.1). It is characterized by the Lie-algebra commutation relations

$$[\hat{\mathbf{x}}_0, \hat{\mathbf{x}}_j] = i\lambda\hat{\mathbf{x}}_j, \quad [\hat{\mathbf{x}}_j, \hat{\mathbf{x}}_k] = 0, \quad j = 1, 2, 3,$$

where the noncommutative parameter² $\lambda \in \mathbb{R}^+$ is usually expected to be of the order of the Planck length.

Such a model was introduced in Ref. [6]-[8] and has been widely studied both from a mathematical and a physical perspective[9, 10].

κ -Minkowski algebra was also proposed in the framework of the Planck scale Physics [11, 12] as a natural candidate for a quantized spacetime in the zero-curvature limit.

Recently, κ -Minkowski gained remarkable attention due to the fact that its symmetries have been proven to be described in terms of a well-known deformation of the Poincaré group, the κ -Poincaré Hopf algebra[13]-[16], which has been derived by contracting the Hopf algebra $SO_q(3, 2)$.

The conserved charges associated with the κ -Poincaré Hopf-algebra transformations have been characterized on the basis of various heuristic arguments [14]. In particular, the identification of the energy-momentum charges with the generators of the κ -Poincaré translations has led to hypothesis that in κ -Minkowski spacetime particles may be submitted to dispersion relations modified with respect to the Einstein ones by the presence of λ -corrections[17, 18]. If the parameter λ is identified with the Planck length, the modification of the dispersion relations would agree with the results of DSR theories which predict the existence of two observer-independent quantities: a velocity scale and a length scale (given by the Planck length)[19, 20].

Because of these implications at the level of fundamental physics, there is a great interest in searching for a robust characterization of the conserved charges associated with the κ -Poincaré symmetry transformations, especially those associated to the κ -Poincaré translations, which would have the meaning of energy and momenta of particles.

The first attempt in this direction seems to have appeared in Ref. [21], where a Noether analysis has been applied to a free scalar field theory in κ -Minkowski spacetime.

²Historically, the noncommutative parameter $\kappa = \lambda^{-1}$ was introduced. This explains the origin of the name “ κ -Minkowski”.

This study solved the issue about the ambiguity among different but equivalent bases of κ -Poincaré by proving that they give rise to the same energy-momentum charges. However, the analysis in Ref. [21] is restricted to the class of four-dimensional differential calculi which are non- κ -Poincaré covariant. In light of this, it seems natural to look for a formulation of the Noether's theorem based on a κ -Poincaré covariant differential calculus. Such a covariant differential calculus is proven to be uniquely defined and coincides with the five-dimensional one constructed by Sitarz, see Ref. [22].

In this paper we plan to apply a generalization of Noether's theorem to the κ -Minkowski translations of a free scalar theory³. In order to do this we introduce the notion of κ -Poincaré covariant translation in κ -Minkowski spacetime based on the five-dimensional differential calculus. By using the five-dimensional vector derivatives we introduce also a Lagrangian which gives rise to κ -Poincaré-invariant equation of motion. By requiring the invariance of the action of the theory under the covariant spacetime translations we obtain *five* conserved charges. It is important to notice that the choice of a κ -Poincaré covariant Lagrangian (in the sense that it produces κ -Poincaré invariant equation of motion) and the use of a covariant differential calculus assure *step by step* the covariance of the formulation of Noether's theorem.

By applying our results to κ -Minkowski *plane waves* we obtain a dispersion relation for the conserved charges. It seems to be interesting that such a dispersion relation looks like the special-relativity (*i.e.* non-Planck-deformed) dispersion relation in which the mass is replaced by the fifth charge⁴. In the case of a massless theory the fifth conserved charge is zero and the κ -Minkowski plane waves satisfy exactly the special-relativity dispersion relation.

In conclusion, the result that we have obtained in this paper seems to disagree with the deformed dispersion relation conjectured on the basis of heuristic arguments and widely used in literature so far.

2 κ -Minkowski Spacetime and κ -Poincaré Hopf-algebra Symmetry

The coordinates of the four-dimensional κ -Minkowski spacetime satisfy the commutation relations of Lie-algebra type

$$[\hat{\mathbf{x}}_0, \hat{\mathbf{x}}_j] = i\lambda \hat{\mathbf{x}}_j, \quad [\hat{\mathbf{x}}_j, \hat{\mathbf{x}}_k] = 0, \quad j, k = 1, 2, 3, \quad (2.2)$$

where $\hat{\mathbf{x}}_0$ has the meaning of time and $\hat{\mathbf{x}}_j$ have the meaning of space coordinates. In the commutative limit $\lambda \rightarrow 0$, κ -Minkowski reduces to the commutative Minkowski spacetime.

The set of coordinates $\hat{\mathbf{x}}_\mu$ and the Lie-algebra relation (2.2) define the associative algebraic structure \mathcal{A}_x of κ -Minkowski. We can consider different bases for the algebra \mathcal{A}_x

³In this paper the analysis of Noether's theorem is restricted to the classical field theory in order to avoid the further complications that arise in quantum field theory, which we postpone to future studies.

⁴A result similar to ours was obtained, independently, by G. Amelino-Camelia *et. al* in Ref. [23].

which are all equivalent (see Refs. [24], [25]). In this paper we shall use the time-to-the-right basis

$$\hat{e}_k = \{e^{-ik\hat{\mathbf{x}}} e^{ik_0\hat{\mathbf{x}}_0}\}, \quad k^\mu \in R^{(1,3)}.$$

where $\hat{\mathbf{x}}_0$ has the meaning of time. The product associated to this basis is

$$\hat{e}_k \hat{e}_p = \hat{e}_{(k \oplus p)},$$

where the non-Abelian sum is $(k \oplus p)_\mu = (k_0 + p_0, k_j + e^{-\lambda k_0} p_j)$.

Notice the following conjugation property

$$e_k^\dagger = e_{\dot{-}k}, \quad (2.3)$$

where $\dot{-}k_\mu$ is called antipode and corresponds to $(-k_0, -e^{\lambda k_0} k_j)$.

We consider fields in κ -Minkowski spacetime as elements of the algebra \mathcal{A}_x

$$\Phi(\hat{\mathbf{x}}) = \int d^4k \mu_\lambda(k) \tilde{\phi}(k) \hat{e}_k, \quad (2.4)$$

where $\tilde{\phi}(k)$ is the Fourier transform of the commutative limit of $\Phi(\hat{\mathbf{x}})$, and $\mu_\lambda(k)$ is an integration measure which is equal to 1 in the commutative limit. The expression of $\Phi(\hat{\mathbf{x}})$ is the generalization of a classical field which is usually represented as a Fourier expansion in plane waves. Here we do not discuss the algebraic properties of the κ -Minkowski functions $\Phi(\hat{\mathbf{x}})$ for which we refer to Ref. [26], where a rigorous analysis has been done on the representations of the κ -Minkowski functions on Hilbert spaces and their C^* -algebra properties.

In Ref. [13] it has been proven that κ -Minkowski noncommutative spacetime is the invariant space⁵ of a quantum deformation of the Poincaré algebra, called κ -Poincaré algebra, originally obtained as a contraction of the Hopf algebra $SO_q(3, 2)$, see Ref. [6]-[8].

The κ -Poincaré Hopf-algebra can be written in a number of different bases. In the so called Majid-Ruegg bicrossproduct basis the generators (P_μ, N_j, M_j) of κ -Poincaré satisfy the following commutation relations

$$\begin{aligned} [P_\mu, P_\nu] &= 0, \\ [M_j, M_k] &= i\varepsilon_{jkl} M_l, \quad [N_j, M_k] = i\varepsilon_{jkl} N_l, \quad [N_j, N_k] = -i\varepsilon_{jkl} M_l, \\ [M_j, P_0] &= 0, \quad [M_j, P_k] = i\varepsilon_{jkl} P_l, \\ [N_j, P_0] &= iP_j, \\ [N_j, P_k] &= i \left[\left(\frac{1 - e^{-2\lambda P_0}}{2\lambda} + \frac{\lambda}{2} \vec{P}^2 \right) \delta_{jk} - \lambda P_j P_k \right], \end{aligned} \quad (2.5)$$

⁵The invariance of κ -Minkowski under κ -Poincaré Hopf-algebra means that the commutation relations of κ -Minkowski are left invariant under the action of the generators of the κ -Poincaré algebra

$$T \triangleright [\hat{\mathbf{x}}_0, \hat{\mathbf{x}}_j] = i\lambda T \triangleright \hat{\mathbf{x}}_j$$

where $T = (P_\mu, N_j, M_j)$. See Ref.[27] for the mathematical definition of action.

and the following co-algebra relations

$$\begin{aligned}\Delta(P_0) &= P_0 \otimes 1 + 1 \otimes P_0 & \Delta(P_j) &= P_j \otimes 1 + e^{-\lambda P_0} \otimes P_j, \\ \Delta(M_j) &= M_j \otimes 1 + 1 \otimes M_j, \\ \Delta(N_j) &= N_j \otimes 1 + e^{-\lambda P_0} \otimes N_j - \lambda \varepsilon_{jkl} P_k \otimes M_l.\end{aligned}\tag{2.6}$$

The “mass-squared” Casimir operator in the Majid-Ruegg bicrossproduct basis takes the form

$$C_\lambda(P) = \frac{2}{\lambda^2} \sinh^2 \left(\frac{\lambda P_0}{2} \right) - e^{\lambda P_0} \vec{P}^2.\tag{2.7}$$

The symmetry generators of the κ -Poincaré Hopf algebra act in the following way on κ -Minkowski functions:

$$\begin{aligned}P_\mu \hat{e}_k &= k_\mu \hat{e}_k, \\ M_j \hat{e}_k &= i \varepsilon_{jlm} k_l \partial_m \hat{e}_k, \\ N_j \hat{e}_k &= i \left(k_j \partial_0 - \left(\frac{1 - e^{-2\lambda k_0}}{2\lambda} + \frac{\lambda}{2} k^2 \right) \partial_j + \lambda k_j k_l \partial_l \right) \hat{e}_k.\end{aligned}\tag{2.8}$$

As the reader can note, in the bicross-product basis the generators of the κ -Lorentz algebra fulfil the commutation relations of the un-deformed Lorentz Lie-algebra. Nevertheless, the symmetry generators of the κ -Poincaré Hopf algebra act in a deformed way on products of functions.

3 The 5D κ -Poincaré-invariant Differential Calculus

The issue of finding differential calculi related to κ -Minkowski spacetime has been investigated in different papers[28, 29]. Sitarz[22] proved that there are no 4D κ -Poincaré-covariant differential calculi, and proposed a 5D differential calculus which is covariant under the left action of the κ -Poincaré Hopf algebra⁶. Then, Gonera *et al.*[31] showed that the lowest dimensional left-covariant calculus for the 4D κ -Minkowski spacetime is uniquely defined and coincides with the 5D calculus proposed by Sitarz.

In the 5D differential calculus the exterior derivative operator d of a generic κ -Minkowski element $\Phi(\hat{\mathbf{x}})$ can be written in terms of vector fields $\mathcal{D}_a(P)$ as follows:

$$\begin{aligned}d\Phi(\hat{\mathbf{x}}) &= dx^a \mathcal{D}_a \Phi(\hat{\mathbf{x}}), \quad a = 0, 1, 2, 3, 4, \\ \mathcal{D}_0(P) &= \frac{i}{\lambda} \left[\sinh(\lambda P_0) + \frac{\lambda^2}{2} e^{\lambda P_0} \vec{P}^2 \right], \\ \mathcal{D}_j(P) &= i P_j e^{\lambda P_0} \quad (j = 1, 2, 3), \\ \mathcal{D}_4(P) &= \frac{1}{\lambda} \left[2 \sinh^2 \left(\frac{\lambda P_0}{2} \right) - \frac{\lambda^2}{2} \vec{P}^2 e^{\lambda P_0} \right],\end{aligned}\tag{3.9}$$

⁶or, equivalently, under the infinitesimal left action of κ -Poincaré quantum group \mathcal{P}_κ , see Ref. [30].

where P_μ are the generators of κ -Poincaré in the bicrossproduct basis and act on κ -Minkowski as in Eq. (2.8) $P_\mu \hat{e}_k = k_\mu \hat{e}_k$. Notice that the last component $\mathcal{D}_4(P)$ coincides with the Casimir operator (2.7).

The 5D differential calculus is obtained in Ref. [22] by the request that the commutation relations (2.2) remain invariant under the action of the κ -Poincaré generators:

$$[dx^a, \hat{\mathbf{x}}^\mu] = \rho_\nu^{a\mu} \hat{\mathbf{x}}^\nu \quad \rightarrow \quad T [dx^a, \hat{\mathbf{x}}^\mu] = \rho_\nu^{a\mu} T \hat{\mathbf{x}}^\nu$$

where T denotes globally the κ -Poincaré generators $(P_\mu; M_j; N_j)$. A differential calculus in which the commutation relations between the 1-form generators and the κ -Minkowski generators remain invariant under the action of symmetry algebra (κ -Poincaré in our case), is called “covariant” differential calculus.

Clearly, the κ -Minkowski derivatives \mathcal{D}_a reduce to the commutative derivatives ∂_μ in the limit $\lambda \rightarrow 0$:

$$\lim_{\lambda \rightarrow 0} \mathcal{D}_\mu(P) = iP_\mu = \partial_\mu, \quad \lim_{\lambda \rightarrow 0} \mathcal{D}_4(P) = 0.$$

The deformed derivatives $\mathcal{D}_a(P)$ have some nice covariance properties. They transform in the classical way under the κ -Poincaré action

$$\begin{aligned} [M_j, \mathcal{D}_\mu] &= i\delta_{\mu k} \epsilon_{jkl} \mathcal{D}_l, \\ [N_j, \mathcal{D}_\mu] &= iD_\mu, \\ [M_j, \mathcal{D}_4] &= [N_j, \mathcal{D}_4] = 0. \end{aligned}$$

Their coproduct can be written as

$$\begin{aligned} \Delta(\mathcal{D}_\beta) &= \mathcal{D}_\beta \otimes e^{\lambda P_0} + e^{-\lambda P_0 \delta_{\beta 0}} \otimes \mathcal{D}_\beta - i\delta_{\beta 0} \lambda e^{-\lambda P_0} \mathcal{D}_j \otimes \mathcal{D}_j, \\ \Delta(\mathcal{D}_4) &= \mathcal{D}_4 \otimes e^{\lambda P_0} + e^{-\lambda P_0} \otimes \mathcal{D}_4 + \frac{1}{\lambda} e^{-\lambda P_0} (e^{\lambda P_0} - 1) \otimes (e^{\lambda P_0} - 1) \\ &\quad + \lambda e^{-\lambda P_0} \mathcal{D}_j \otimes \mathcal{D}_j. \end{aligned} \tag{3.10}$$

The one-form generators $dx^a = (dx^\mu, dx^4)$ satisfy the commutation relations (see Ref. [22])

$$\begin{aligned} [dx^\mu, \hat{\mathbf{x}}^\nu] &= i\lambda(g^{0\mu} dx^\nu - g^{\mu\nu} dx^0 + ig^{\mu\nu} dx^4), \\ [dx^4, \hat{\mathbf{x}}^\mu] &= \lambda dx^\mu. \end{aligned} \tag{3.11}$$

The fifth one-form generator is here denoted by dx^4 , but this is of course only a formal notation since there is no fifth κ -Minkowski coordinate $\hat{\mathbf{x}}^4$.

The one-forms dx^a have the following hermitian properties:

$$(dx^\mu)^* = dx^\mu, \quad (dx^4)^* = -dx^4.$$

The external algebra takes the standard form

$$dx^a \wedge dx^b = -dx^b \wedge dx^a \quad (a, b = 0, 1, 2, 3, 4),$$

and

$$d(dx^\mu) = 0, \quad d(dx^4) = -\frac{2i}{\lambda} d(dx^\mu) \wedge d(dx_\mu).$$

More details about the 5D differential calculus can be found in Ref. [32] and references therein. For the purpose of this article we quote the following commutation relation between one forms dx_a and κ -Minkowski functions

$$\Phi dx_b \mathcal{D}^b \Psi = dx^b [\mathcal{D}_b(\Phi \Psi) - (\mathcal{D}_b \Phi) \Psi]. \quad (3.12)$$

They can be easily proven by the iterative use of Eq. (3.11).

4 Noether's Theorem for Translations in κ -Minkowski Spacetime

The formulation of Noether's theorem in commutative spacetime is recalled in Appendix A. In this section we shall generalize Noether's theorem for translations in κ -Minkowski.

We consider the infinitesimal translation in κ -Minkowski

$$\hat{\mathbf{x}}_\mu \rightarrow \hat{\mathbf{x}}'_\mu = \hat{\mathbf{x}}_\mu + dx_\mu, \quad (4.13)$$

where dx_μ are the infinitesimal displacements. In order to ensure that the point $\hat{\mathbf{x}}'_\mu$ still belongs to the κ -Minkowski spacetime

$$[\hat{\mathbf{x}}'_0, \hat{\mathbf{x}}'_j] = i\lambda \hat{\mathbf{x}}'_j, \quad [\hat{\mathbf{x}}_j, \hat{\mathbf{x}}_k] = 0$$

the translation parameters dx_μ must satisfy non-zero commutation relations with the κ -Minkowski generators $\hat{\mathbf{x}}_\mu$, namely

$$[\hat{\mathbf{x}}_0, dx_j] + [dx_0, \hat{\mathbf{x}}_j] = i\lambda dx_j$$

$$[\hat{\mathbf{x}}_j, dx_k] + [dx_j, \hat{\mathbf{x}}_k] = 0$$

There are different choices for the commutation relations $[\hat{\mathbf{x}}_\mu, dx_\nu]$ which fulfill the equations above. The various choices select different choices of differential calculi in κ -Minkowski (see, for example, Ref. [21] where the infinitesimal displacement is denoted by ε_μ). In this paper we chose the 5-dimensional differential calculus (3.11).

As the spacetime changes also the field changes in order to preserve the relativistic invariance. The total variation can be written as the sum of two terms⁷

$$\delta_T \Phi(\hat{\mathbf{x}}) = \Phi'(\hat{\mathbf{x}}') - \Phi(\hat{\mathbf{x}}) \approx \delta \Phi(\hat{\mathbf{x}}) + dx^a \mathcal{D}_a \Phi, \quad (4.14)$$

where $\delta \Phi(\hat{\mathbf{x}})$ is the contribution of the change of the form of the field and $dx^a \mathcal{D}_a \Phi$ is the contribution of the change of the argument $\hat{\mathbf{x}}$.

The first step for generalizing the results of Appendix A to κ -Minkowski consists in introducing an action for free scalar particles in κ -Minkowski. In order to do this we need to introduce a Lagrangian density and define an integration map.

⁷we consider the approximation $\delta \Phi(\hat{\mathbf{x}}') = \delta \Phi(\hat{\mathbf{x}})$ at the first order in ε .

A good candidate for a covariant generalization of the Lagrangian for scalar particles is

$$\mathcal{L}(\hat{\mathbf{x}}) = \frac{1}{2} \left(\tilde{\mathcal{D}}_a \Phi(\hat{\mathbf{x}}) \cdot \mathcal{D}^a \Phi(\hat{\mathbf{x}}) - m^2 \Phi^2(\hat{\mathbf{x}}) \right) \quad (4.15)$$

with $\tilde{\mathcal{D}}^a \equiv \frac{i}{\lambda} \left(\sinh(\lambda P_0) - \frac{\lambda^2}{2} e^{\lambda P_0} \vec{P}^2, \lambda P^j, i(2 \sinh^2(\frac{\lambda P_0}{2}) - \frac{\lambda^2}{2} e^{\lambda P_0} \vec{P}^2) \right)$. In the next section we will see that the Lagrangian (4.15) gives rise to κ -Poincaré covariant equation of motion (EoM). The choice of a Lagrangian which produces κ -Poincaré-covariant EoM is fundamental to assure a κ -Poincaré-covariant formulation of Noether theorem.

Concerning the integration map, a natural κ -Poincaré translation-invariant choice is (see Refs. [33], [34])

$$\int \Phi(\hat{\mathbf{x}}) = \int d^4x \phi(x), \quad (4.16)$$

where the κ -Minkowski function $\Phi(\hat{\mathbf{x}})$ is written in the time-to-the-right ordering (*i.e.* in terms of the time-to-the-right basis \hat{e}_k , see Eq. (2.4)) and the right side is the usual integration of the underlying commutative function $\phi(x)$. This prescription is such that the integral of a partial derivative of a suitably decaying function ϕ vanishes.

The next four properties of the integration we have considered will be of precious help.

- The integral of an element of the κ -Minkowski basis is a standard Dirac function

$$\int \hat{e}_k = \delta(k). \quad (4.17)$$

- The following cyclicity property holds

$$\int [\Phi \Psi + \Psi \Phi] = \int [(1 + e^{3\lambda P_0}) \Phi] \Psi. \quad (4.18)$$

- The following integrations by parts hold⁸

$$\int \tilde{\mathcal{D}}^a \Phi \cdot \Psi = - \int \Phi \cdot \mathcal{D}^a \Psi, \quad (4.19)$$

$$\int \Phi \cdot \mathcal{D}^a \mathcal{D}_a \Psi = \int \mathcal{D}^a \mathcal{D}_a \Phi \cdot \Psi, \quad (4.20)$$

where $\tilde{\mathcal{D}}^a = \frac{i}{\lambda} \left(\sinh(\lambda P_0) - \frac{\lambda^2}{2} e^{\lambda P_0} \vec{P}^2, \lambda P^j, i(2 \sinh^2(\frac{\lambda P_0}{2}) - \frac{\lambda^2}{2} e^{\lambda P_0} \vec{P}^2) \right)$.

- The space integral of a divergence $\int d^3\hat{\mathbf{x}} P_j \Psi(\hat{\mathbf{x}})$ is zero:

$$\int d^3 P_j(\Psi(\hat{\mathbf{x}})) = \int d^4k \psi(k) \left(\int d^3 P_j e^{ik\hat{\mathbf{x}}} \right) e^{-ik_0\hat{\mathbf{x}}_0} = \int d^4k \psi(k) k_j \delta^3(k) e^{-ik_0\hat{\mathbf{x}}_0} = 0. \quad (4.21)$$

⁸One can prove it by using the following rule of integration by parts

$$0 = \int P_\mu(\Phi \Phi) = \int \Delta(P_\mu) \cdot (\Phi \otimes \Phi) = \int (P_\mu \Phi) \Phi + \int (e^{-\lambda P_0 \delta_{\mu j}} \Phi)(P_\mu \Phi).$$

The invariance under translation can be easily proven⁹ by using property (i)

$$\int [\Phi'(\hat{\mathbf{x}}) - \Phi(\hat{\mathbf{x}})] = -dx^a \int \mathcal{D}_a \Phi(\hat{\mathbf{x}}) = -dx^a \tilde{\phi}(p) \mathcal{D}_a(p) \delta(p) = 0.$$

The action of the theory is obtained by integrating the Lagrangian density (4.15)

$$S[\Phi] = \int \mathcal{L}(\hat{\mathbf{x}}) = \frac{1}{2} \int \left(\tilde{\mathcal{D}}_a \Phi(\hat{\mathbf{x}}) \cdot \mathcal{D}^a \Phi(\hat{\mathbf{x}}) - m^2 \Phi^2(\hat{\mathbf{x}}) \right)$$

with the integral map defined in Eq. (4.16) and the operator $\tilde{\mathcal{D}}$ introduced in Eq. (4.19).

Next, we get the EoM for free scalar particles in κ -Minkowski. EoM are used to obtain the conserved currents for *on-shell* particles.

4.1 κ -Poincaré-covariant Equations of Motion in κ -Minkowski

In this section we show that the Lagrangian (4.15) gives rise to κ -covariant equation of motion (EoM) in κ -Minkowski. In this way we intend the *kappa*-covariance of the Lagrangian.

Let us consider an arbitrary variation $\delta\Phi$. Hamilton's principle states that

$$\delta S[\Phi] = \frac{1}{2} \int [(\tilde{\mathcal{D}}_a \Phi) \mathcal{D}^a \delta\Phi + (\tilde{\mathcal{D}}_a \delta\Phi) \mathcal{D}^a \Phi - m^2 \Phi \cdot \delta\Phi - m^2 \delta\Phi \cdot \Phi] = 0. \quad (4.22)$$

By using properties (4.19), (4.20) and (4.18), we obtain the EoM:

$$\begin{aligned} 0 &= \int \Phi[(\mathcal{D}_a \mathcal{D}^a + m^2) \delta\Phi] + \delta\Phi(\mathcal{D}_a \mathcal{D}^a \Phi + m^2) \\ &= \int (\mathcal{D}_a \mathcal{D}^a + m^2) \Phi \cdot \delta\Phi + \delta\Phi \cdot (\mathcal{D}_a \mathcal{D}^a + m^2) \Phi \\ &= \int (1 + e^{3\lambda P_0})(\mathcal{D}_a \mathcal{D}^a + m^2) \Phi \cdot \delta\Phi \rightarrow (\mathcal{D}_a \mathcal{D}^a + m^2) \Phi_0 = 0, \end{aligned} \quad (4.23)$$

which contains a covariant generalization of D'Alembert's operator of order two in the generalized derivatives \mathcal{D}_a

$$\square_\lambda = \mathcal{D}_a \mathcal{D}^a = -\left(\frac{4}{\lambda^2} \sinh^2(\lambda P_0/2) - \vec{P}^2 e^{\lambda P_0}\right), \quad (4.24)$$

which coincides with the Casimir operator (2.7) and turns out to be covariant (under the generators T of κ -Poincaré)

$$[T, \mathcal{D}_a \mathcal{D}^a] = 0.$$

Of course, in the commutative limit, \square_λ reduces to the commutative D'Alembert operator

$$\mathcal{D}_a \mathcal{D}^a \rightarrow \partial_\mu \partial^\mu.$$

Thus the Lagrangian (4.15) gives rise to EoM invariant under the action of the κ -Poincaré generators.

⁹Observe that the step $\int dx^a f(\hat{\mathbf{x}}) = dx^a \int f(\hat{\mathbf{x}})$ is well defined even though dx^a does not commute with $f(\hat{\mathbf{x}})$: one can show indeed (see Appendix C in Ref. [32]) that $[dx^a, f(\hat{\mathbf{x}})] = O(P_\mu)$ thus, for any decaying function f , $\int [dx^a, f(\hat{\mathbf{x}})] = 0$ and the integration map can be applied directly to the function $f(\hat{\mathbf{x}})$.

4.2 Conserved charges for κ -Minkowski translations

In this section we obtain the conserved currents for translations in κ -Minkowski. We show here the main steps of the procedure. The details of the calculation can be found in Appendix B.

As in the commutative case, the symmetry condition of the action can be formulated as

$$\delta S_\Omega[\Phi] = \frac{1}{2} \int_\Omega [\mathcal{L}(\Phi'(\hat{\mathbf{x}}')) - \mathcal{L}(\Phi(\hat{\mathbf{x}}))] = 0.$$

According to the variation of Φ , Eq. (4.14), the variation of the action has two contributions

$$\delta S_\Omega[\Phi] = \frac{1}{2} \int_\Omega [\delta \mathcal{L} + dx_a \mathcal{D}^a \mathcal{L}].$$

Considering the variation of the action on the EoM, and after some calculations, we can put the variation in the form (see appendix for details)

$$\delta S_\Omega[\Phi] = dx^b \int_\Omega D^\mu J_{\mu b}.$$

The invariance with respect to spacetime translations gives rise to the continuity equation

$$D^\mu(P) J_{\mu b} = 0, \quad (4.25)$$

where $D^\mu =$ is a function of momenta and $J_{\mu b}$ is a five-dimensional generalization of the energy-momentum tensor.

The nutshell κ -Minkowski-particles¹⁰ can be written as

$$\Phi_0 = \int d^4k \tilde{\phi}(k) \hat{e}_k \delta(C_\lambda(k)), \quad (4.26)$$

where $C_\lambda(k) = \frac{4}{\lambda^2} \sinh^2(\frac{\lambda k_0}{2}) - e^{\lambda k_0} k^2 - m^2$.

The explicit form of $J_{\mu b}$ for on-shell particle Φ_0 is obtained in Appendix C and corresponds to

$$\begin{aligned} J_{\mu\beta} &= e^{-\lambda P_0} \mathcal{D}_\beta \Phi_0 \cdot D_\mu \Phi_0 + e^{-\lambda P_0 \delta_{\beta 0}} \tilde{D}_\mu \Phi_0 \cdot \mathcal{D}_\beta \Phi_0 - i \delta_{\beta 0} \lambda \tilde{D}_j \tilde{D}_\mu \Phi_0 \cdot D_j \Phi_0 + \\ &\quad + i g_{\mu 0} \lambda m^2 e^{-\lambda P_0} \mathcal{D}_\beta \Phi_0 \cdot e^{\lambda P_0} \Phi_0 - g_{\mu\beta} (1 + e^{-\lambda P_0})^{\delta_{\beta 0}} \mathcal{L}_{\Phi_0} \\ J_{\mu 4} &= e^{-\lambda P_0} \mathcal{D}_4 \Phi_0 \cdot D_\mu \Phi_0 + e^{-\lambda P_0} \tilde{D}_\mu \Phi_0 \cdot \mathcal{D}_4 \Phi_0 - \lambda \tilde{D}_\mu \tilde{D}^\nu \Phi_0 \cdot D_\nu \Phi_0 + \\ &\quad + i g_{\mu 0} \lambda m^2 e^{-\lambda P_0} \mathcal{D}_4 \Phi_0 \cdot e^{\lambda P_0} \Phi_0 + g_{\mu 0} \lambda \tilde{D}_0 \mathcal{L}_{\Phi_0} \end{aligned}$$

By spacial integration of $D^\mu J_{\mu b}$ we obtain

$$D^0 \int d^3 J_{0b} = \int d^3 D_j J_{jb} = 0, \quad (4.27)$$

¹⁰We call nutshell κ -Minkowski-particles the free scalar particles Φ_0 which satisfy the EoM in κ -Minkowski spacetime ($\square_\lambda + m^2$) $\Phi_0 = 0$.

where we used the property that the space integral of a divergence is zero, see Eq. (4.21).

Thus J_{0b} are the conserved currents which we are looking for. On the EoM, the components of the current J_{0b} are (see the detailed computation in Appendix C)

$$\begin{aligned} J_{00} &= \tilde{D}_0 \Phi \cdot D_0 \Phi + \tilde{D}_j \Phi \cdot D_j \Phi + m^2 e^{-\lambda P_0} \Phi \cdot e^{\lambda P_0} \Phi - i J_{04} \\ J_{0j} &= \tilde{D}_j \Phi \cdot D_0 \Phi + \tilde{D}_0 \Phi \cdot D_j \Phi + i \lambda m^2 \tilde{D}_j \Phi \cdot e^{\lambda P_0} \Phi \\ J_{04} &= \frac{\lambda m^2}{2} (\tilde{D}_0 + i \lambda m^2) (e^{-\lambda P_0} \Phi \cdot e^{\lambda P_0} \Phi) - \lambda \tilde{D}_0 \tilde{D}^\nu \Phi \cdot D_\nu \Phi + \lambda \tilde{D}_0 \mathcal{L}. \end{aligned}$$

Eq. (4.27) implies that the quantity $Q_a = \int d^3x J_{0a}$, corresponding to the conserved charges, is $\hat{\mathbf{x}}_0$ -independent. We show the time-independence in Appendix D, where an explicit construction of the conserved charges is obtained. The explicit form of the conserved charges is

$$Q_b = -\frac{i}{2} \int d^4p \tilde{\phi}(\dot{-}p) \tilde{\phi}(p) e^{2\lambda p_0} \tilde{\mathcal{D}}_b(p) \operatorname{sgn} \left(\frac{2}{\lambda^2} (1 - e^{-\lambda p_0}) + m^2 \right) \delta[C_\lambda(p)] \quad (4.28)$$

where $\mathcal{D}_b(p)$ is defined as in the Eq. (B.3) and $\operatorname{sgn}(y) = \frac{y}{|y|}$ is the sign function. This form shows the time-independence of the charges Q_b .

In this way we have obtained *five* conserved-charges on the equations of motion. In the limit $m = 0$ the fifth charge Q_4 is zero for on-shell massless particles, and we have four-conserved charges.

In the next section we show that the analysis in terms of plane waves allows us to recognize a relation between conserved charges. This relation can have the meaning of dispersion relation for particles and it allows to identify the energy-momentum vector.

4.3 Plane waves and energy-momentum conservation law

In Appendix E we have proved that the Fourier transform of a real on-shell *plane wave* in κ -Minkowski is described by the function

$$\tilde{\phi}_p(k) = \delta^{(3)}(k - p) H \left[k_0 - \ln \left(1 + \frac{\lambda^2 m^2}{2} \right) \right] + \delta^{(3)}(k + p) H \left[-k_0 - \ln \left(1 + \frac{\lambda^2 m^2}{2} \right) \right] \quad (4.29)$$

and we have also proved that

$$\tilde{\phi}_p(\dot{-}k) = e^{-3\lambda k_0} \tilde{\phi}_p(k). \quad (4.30)$$

By substituting Eq. (4.29) and Eq. (4.30) in Eq. (4.28), we can compute the five-

conserved charges

$$\begin{aligned}
Q_b &= -\frac{i}{2} \int d^4k \tilde{\phi}(-k) \tilde{\phi}(k) e^{-\lambda k_0} \tilde{\mathcal{D}}_b(k_0, \vec{k}) \operatorname{sgn} \left(\frac{2}{\lambda^2} (1 - e^{-\lambda p_0}) + m^2 \right) \delta[C_\lambda(k)] \\
&= -i \int d^4k e^{-\lambda k_0} \left[\delta^{(3)}(k - p) H(k_0 - \ln(1 + \frac{\lambda^2 m^2}{2})) \right]^2 \tilde{\mathcal{D}}_b(k_0, \vec{k}) \cdot \\
&\quad \cdot \operatorname{sgn} \left(\frac{2}{\lambda^2} (1 - e^{-\lambda p_0}) + m^2 \right) \delta[C_\lambda(k)] \\
&= -iV \frac{\lambda}{2} \int dk_0 e^{-\lambda k_0} \tilde{\mathcal{D}}_b(k_0, \vec{p}) \frac{\delta[k_0 - w_+(p)]}{|1 + \frac{\lambda^2 m^2}{2} - e^{-\lambda k_0}|} \\
&= \frac{-i \frac{\lambda V}{2} e^{-\lambda w_+(p)}}{|1 + \frac{\lambda^2 m^2}{2} - e^{-\lambda w_+(p)}|} \tilde{\mathcal{D}}_b(w_+(p), \vec{p}).
\end{aligned}$$

Thus, the conserved charges are proportional to $\tilde{\mathcal{D}}_b(w_+(p), \vec{p})$: $Q_b \sim \tilde{\mathcal{D}}_b(w_+(p), \vec{p})$ and, because of the following relation

$$\begin{aligned}
[\tilde{\mathcal{D}}_\mu \tilde{\mathcal{D}}^\mu](w_+(p), \vec{p}) &= [\tilde{D}_0(w_+(p), \vec{p}) + i \frac{\lambda m^2}{2}]^2 - \tilde{D}_j^2 \\
&= \tilde{D}_\mu \tilde{D}^\mu + i \lambda m^2 \tilde{D}_0 - \frac{\lambda^2 m^4}{4} = -e^{-\lambda w_+(p)} m^2 + i \lambda m^2 \tilde{D}_0 - \frac{\lambda^2 m^4}{4} \\
&= -e^{-\lambda w_+(p)} m^2 - m^2 (1 - e^{-\lambda w_+(p)}) - \frac{\lambda^2 m^4}{4} = -(\frac{4}{\lambda^2 m^2} + 1) \frac{\lambda^2 m^2}{4} \\
&= -\left(\frac{4}{\lambda^2 m^2} + 1 \right) \tilde{\mathcal{D}}_4^2(w_+(p), \vec{p}),
\end{aligned}$$

the relation among the charges is¹¹

$$Q_\mu Q^\mu + (1 + \frac{4}{\lambda^2 m^2}) Q_4^2 = 0. \quad (4.31)$$

Notice that in the massless case ($m = 0$) the relation (4.31) reads as

$$Q_\mu Q^\mu = 0$$

Thus, if we give Q_μ the meaning of the energy-momentum vector, the relation above says that the dispersion relation for massless Klein-Gordon particles in κ -Minkowski spacetime is not-deformed and coincides with its special-relativistic limit.

In the case of massive particles a fifth conserved charges (Q_4) appears. If we define $\hat{Q}_b = i^{\delta_{b4}} (1 + \frac{4}{\lambda^2 m^2})^{\frac{\delta_{b4}}{2}} Q_b$, all the charges are real and we get the relation

$$\hat{Q}_\mu \hat{Q}^\mu = \hat{Q}_4^2 \quad (4.32)$$

which might be viewed as a special-relativistic dispersion-relation with a generalized mass term represented by \hat{Q}_4 .

¹¹This result has been also obtained, independently, by G. Amelino-Camelia *et al.* in Ref. [23].

5 Comparison with Previous Results

To our knowledge, the first exploratory analysis of Noether's theorem in noncommutative spacetime appeared in ref. [21]. In this paper the study of translation symmetries has been applied to the example of κ -Minkowski noncommutative spacetime. According to some algebraic arguments, the symmetries of κ -Minkowski should be described in terms of a Plank-scale-deformation of the Poincaré algebra, and a Planck-scale-deformation should affect the particle dispersion relations. The paper Ref. [21] was aimed at establishing whether these formal observations about the presence of nonclassical symmetries might have been confirmed by a physical perspective based on Noether's analysis.

By considering a much used (four dimensional) differential calculus in κ -Minkowski spacetime, the presence of some nonlinearity in the energy-momentum relation for scalar fields did emerge in Ref. [21]. For massless particles it has been obtained the following dispersion relations

$$\frac{4}{\lambda^2} \sinh^2 \frac{\lambda E}{2} - e^{\lambda E} \sum_{i=1}^3 P_i^2 = 0, \quad (5.33)$$

where E, P_i are the energy and momenta of the physical particle.

However it is natural to wonder if this result might depend on the choice of the differential calculus and to search for an alternative formulation based on a different calculus.

The present paper has generalized the study of Ref. [21] by replacing the four dimensional differential calculus by the five dimensional κ -Poincaré covariant differential calculus. This replacement should guarantee a κ -Poincaré-covariant formulation of Noether's theorem. As in Ref. [21], our study has been focused on translations for a scalar field theory in κ -Minkowski spacetime. Our analysis has revealed that the relation (5.33) does depend on the choice of the differential calculus. Indeed, using a κ -Poincaré-covariant differential calculus we have found classical properties for the energy-momentum charges which, in the massless case, turn out to satisfy the special-relativity relation

$$E^2 - \sum_{i=1}^3 P_i^2 = 0.$$

6 Conclusions

In this paper we have constructed a κ -Poincaré covariant formulation of Noether's theorem for translations in κ -Minkowski. In order to guarantee the κ -Poincaré covariance of the formulation we have based our analysis on the five dimensional covariant differential calculus. We have obtain exactly the special-relativistic dispersion relations for massless free scalar particles, while for massive particles we have found a dispersion relation similar to the special-relativistic one but with a further (fifth) conserved charge which might have the meaning of a generalized mass.

Our results does not agree with the κ -Minkowski deformed dispersion relation conjectured on the basis of some heuristic arguments and widely used in literature. However, the

fact that in this first exploratory application of our description of symmetries we have only considered a free scalar theory in κ -Minkowski spacetime might be a significant limitation. One may think that the theory considered here does not have enough structure to give proper physical significance to energy-momentum; such hypothesis may deserve attention for future investigations. It would be of interest to explore whether massless particles in κ -Poincaré spacetime be affected by a Planck-scale-deformation of the special-relativistic dispersion relations if one attempts to extend our result to the case of more structured theories, such as interacting theories or gauge theories.

We remark that in this paper only translations are considered. In principle, the same method we have used for translations can be applied to the Lorentz transformations. Future studies may wish to investigate Lorentz transformations in κ -Minkowski and the construction of conserved angular momentum (if it exist).

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A Noether Theorem in Commutative Spacetime

Classical Noether's theorem states that symmetry properties of the Lagrangian (or Hamiltonian) imply the existence of conserved quantities (see for example Ref. [35]).

In order to formulate Noether's theorem we need the equation of motion (EoM). For the sake of simplicity, we consider the case of a scalar field $\phi(x)$.

Let us introduce the Lagrangian density $\mathcal{L}(\phi, \partial_\mu \phi, x_\mu)$ which is, in general, a function of the fields ϕ as well of the field derivatives $\partial_\mu \phi$, and in general, might well be an explicit function of x_μ .

We derive the EoM through Hamilton's principle by variation of the action, *i.e.* the integral of $\mathcal{L}(\phi, \partial_\mu \phi, x_\mu)$, over a region in four-space

$$S = \int_{\Omega} d^4x \mathcal{L}(\phi, \partial_\mu \phi, x_\mu). \quad (\text{A.1})$$

We consider an arbitrary variation on ϕ and $\partial_\mu \phi$, which are taken to be zero at the bounding surface $\Gamma(\Omega)$. Hamilton's principle states that

$$\delta S = \int_{\Omega} d^4x \delta \mathcal{L} = 0.$$

The variation $\delta \mathcal{L}$ of the Lagrangian can be written as

$$\delta \mathcal{L}(\phi, \partial_\mu \phi, x_\mu) = \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \delta \partial_\mu \phi,$$

or equivalently, by using the linearity of δ with respect to the derivative $\delta \partial_\mu \phi = \partial_\mu \delta \phi$,

$$\delta \mathcal{L}(\phi, \partial_\mu \phi, x_\mu) = \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial_\mu \delta \phi. \quad (\text{A.2})$$

Thus

$$\begin{aligned}
\delta S &= \int_{\Omega} d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi)} \partial_{\mu} \delta \phi \right\} \\
&= \int_{\Omega} d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} - \partial_{\mu} \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi)} \right\} \delta \phi + \int_{\Omega} d^4x \partial_{\mu} \left\{ \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi)} \delta \phi \right\} \\
&= \int_{\Omega} d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} - \partial_{\mu} \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi)} \right\} \delta \phi = 0,
\end{aligned} \tag{A.3}$$

where the second integral in Eq. (A.3) can be transformed by a four-dimensional divergence theorem into an integral over the surface $\Gamma(\Omega)$ bounding the region Ω and then vanishes since $\delta \phi|_{\Gamma(\Omega)} = 0$.

By requiring that $\delta S = 0$ for any arbitrary variation $\delta \phi$, we get the EoM

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dx_{\mu}} \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi)} \tag{A.4}$$

which is satisfied by on-shell particles.

We now recall the formulation of Noether's theorem for on-shell particles, which is useful in order to extend the construction to the noncommutative case.

Noether's theorem applies to continuous transformations, and here we are dealing only with them. Symmetry under coordinate transformation refers to the effects of an infinitesimal transformation of the form

$$x_{\mu} \rightarrow x'_{\mu} = x_{\mu} + \delta x_{\mu},$$

where the infinitesimal change δx_{μ} may be a function of all the other coordinates x_{ν} . The effect of a transformation in the fields themselves may be described by

$$\phi(x_{\mu}) \rightarrow \phi'(x'_{\mu}) = \phi(x_{\mu}) + \delta_T \phi(x_{\mu}),$$

where the total variation $\delta_T \phi(x)$ results from changes of both the form and the argument of the function $\phi(x)$.

As a consequence of the transformations of both the coordinates and fields, the Lagrangian appears, in general, as a different function of both the spacetime coordinates and the fields

$$\mathcal{L}(\phi, \partial_{\mu} \phi, x_{\mu}) \rightarrow \mathcal{L}'(\phi'(x'), \partial_{\mu} \phi'(x'), x'_{\mu}).$$

The symmetry or invariance condition of the Lagrangian, can be generalized at the level of the action integral, so that the invariance of the magnitude of the action integral under the transformation leads to the existence of conserved quantities

$$\delta_T S[\phi] = \int_{\Omega'} d^4x \mathcal{L}'(\phi'(x), \partial_{\mu} \phi'(x), x) - \int_{\Omega} d^4x \mathcal{L}(\phi, \partial_{\mu} \phi, x) = 0,$$

where Ω is an arbitrary region in the 4-D spacetime, and Ω' its transformation.

By means of some computations we get

$$0 = \int_{\Omega} d^4x \{ \mathcal{L}(\phi'(x)) - \mathcal{L}(\phi(x)) + \partial_{\mu}[\mathcal{L}\delta x^{\mu}] \}.$$

The variation $\delta_T S$ has two contributions: the integral of $\delta_{\phi}\mathcal{L}(\phi)$ is the variation of the action due to the variation of the functional form of the field¹², while the integral of $\partial_{\mu}[\mathcal{L}\delta x^{\mu}]$ comes from the variation in the four-volume $\delta\Omega$

$$0 = \int_{\Omega} d^4x \{ \delta_{\phi}\mathcal{L}(\phi) + \partial_{\mu}[\mathcal{L}\delta x^{\mu}] \}. \quad (\text{A.5})$$

Writing the variation $\delta_{\phi}\mathcal{L}(\phi)$ as in Eq. (A.2) and integrating by part, we get

$$0 = \int_{\Omega} d^4x \left\{ \left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_{\mu}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)} \right) \delta\phi + \partial_{\mu} \left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)} \delta\phi + \mathcal{L}\delta x^{\mu} \right) \right\}.$$

The first integral is zero for on-shell particles because of EoM Eq. (A.4), and the formulation of the invariance reads

$$0 = \int_{\Omega} d^4x \frac{d}{dx_{\mu}} \left\{ \frac{\partial\mathcal{L}}{\partial\phi_{,\mu}} \delta\phi + \mathcal{L}\delta x_{\mu} \right\}, \quad \phi \text{ on-shell.} \quad (\text{A.6})$$

Since the previous equality holds for any Ω , we have the continuity equation for the current $J_{\mu} = \frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}\delta\phi + \mathcal{L}\delta x_{\mu}$

$$\frac{d}{dx_{\mu}} \left\{ \frac{\partial\mathcal{L}}{\partial\phi_{,\mu}} \delta\phi + \mathcal{L}\delta x_{\mu} \right\} = 0.$$

The continuity equation tells us that if we integrate this current over a space-like slice, we get a conserved quantity called the Noether charge.

As a consequence of Noether's theorem the invariance of physical systems with respect to the 10-Poincaré transformations gives the law of conservation of 10-quantities. Here we focus only on translations. The invariance with respect to spacetime translations gives the well known law of conservation of energy-momentum.

A finite translation $x_{\mu} \rightarrow x'_{\mu} = x_{\mu} + a_{\mu}$ induces the following field transformation

$$\phi'(x) = \phi + \delta\phi = \phi(x) - a^{\mu}\partial_{\mu}\phi(x).$$

Noether's theorem states then the existence of a energy-momentum tensor $T_{\mu\nu}$ which satisfies the continuity equation

$$0 = \partial^{\mu}T_{\mu\nu} = \frac{d}{dx_{\mu}} \left\{ \frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}\phi_{,\nu} - g_{\mu\nu}\mathcal{L} \right\}.$$

This equation describes the conservation of *four* energy-momentum charges

$$Q_{\mu} = \int d^3x T_{0\mu} = \int d^3x \left\{ \frac{\partial\mathcal{L}}{\partial\phi_{,0}}\phi_{,\nu} - g_{0\nu}\mathcal{L} \right\}.$$

¹²which affects also the derivatives of the fields $\phi_{,\mu}$.

It is easy to show that the energy-momentum tensor for a real, free scalar field $\phi(x)$ described by a Lagrangian density $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi$ is of the form

$$T_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}\phi_{,\nu} - g_{\mu\nu}\mathcal{L} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L}. \quad (\text{A.7})$$

In the following section, we shall construct a formulation of Noether's theorem in κ -Minkowski which provides us a generalization of this result.

B Properties of the 5D Vector Fields on the Solutions of the EoM

We notice that the deformed derivatives take the following form on the solutions of the EoM ($\Phi_0 : (\Box_\lambda + m^2)\Phi_0 = 0$)

$$\begin{aligned} \mathcal{D}_a\Phi_0 &= \left\{ \frac{i}{\lambda} \left[e^{\lambda P_0} - \left(1 + \frac{\lambda^2}{2}m^2\right) \right], iP_j e^{\lambda P_0}, \frac{\lambda}{2}m^2 \right\} \Phi_0, \\ \tilde{\mathcal{D}}_a\Phi_0 &= \left\{ -\frac{i}{\lambda} \left[e^{-\lambda P_0} - \left(1 + \frac{\lambda^2}{2}m^2\right) \right], iP_j, -\frac{\lambda}{2}m^2 \right\} \Phi_0. \end{aligned} \quad (\text{B.1})$$

It is useful to introduce the following vector fields (which correspond to Eq. (B.1) in the limit $m = 0$)

$$\begin{aligned} D_a &= \left\{ \frac{i}{\lambda} [e^{\lambda P_0} - 1], ie^{\lambda P_0}P_j, 0 \right\} \\ \tilde{D}_a &= \left\{ \frac{i}{\lambda} e^{-\lambda P_0} [e^{\lambda P_0} - 1], iP_j, 0 \right\} = e^{-\lambda P_0} D_a \end{aligned} \quad (\text{B.2})$$

The following relations hold

$$\begin{aligned} \mathcal{D}_a\Phi_0 &= \left[D_0 - i\frac{\lambda}{2}m^2, D_j, \frac{\lambda}{2}m^2 \right] \Phi_0 \\ \tilde{\mathcal{D}}_a\Phi_0 &= \left[\tilde{D}_0 + i\frac{\lambda}{2}m^2, \tilde{D}_j, -\frac{\lambda}{2}m^2 \right] \Phi_0 \end{aligned} \quad (\text{B.3})$$

It is easy to find the coproduct of D_μ and \tilde{D}_μ on Φ_0

$$\Delta(D)_\mu = D_\mu \otimes e^{\lambda P_0} + 1 \otimes D_\mu. \quad (\text{B.4})$$

$$\Delta(\tilde{D}_\mu) = \tilde{D}_\mu \otimes 1 + e^{-\lambda P_0} \otimes \tilde{D}_\mu. \quad (\text{B.5})$$

Observe that the action for on-shell particles can be written as

$$\begin{aligned} S[\Phi_0] &= \frac{1}{2} \int [\tilde{D}^a\Phi_0 \cdot \mathcal{D}_a\Phi_0 - m^2\Phi_0^2] \\ &= \frac{1}{2} \int [\tilde{D}^\mu\Phi_0 D_\mu\Phi_0 - m^2\Phi_0 e^{\lambda P_0}\Phi_0] \end{aligned} \quad (\text{B.6})$$

C κ -Deformed Conserved Currents

Let us consider the infinitesimal translation in κ -Minkowski spacetime

$$\hat{\mathbf{x}}_\mu \rightarrow \hat{\mathbf{x}}'_\mu = \hat{\mathbf{x}}_\mu + dx_\mu \quad (\text{C.1})$$

where dx_μ is the infinitesimal displacement.

The total variation $\delta_T \Phi(\hat{\mathbf{x}})$ of the field $\Phi(\hat{\mathbf{x}})$ under Eq. (C.1) can be written as

$$\begin{aligned} \delta_T \Phi(\hat{\mathbf{x}}) &= \Phi'(\hat{\mathbf{x}}') - \Phi(\hat{\mathbf{x}}) = [\Phi'(\hat{\mathbf{x}}') - \Phi(\hat{\mathbf{x}}')] + [\Phi(\hat{\mathbf{x}}') - \Phi(\hat{\mathbf{x}})] \\ &= \delta\Phi(\hat{\mathbf{x}}') + d\Phi \\ &\approx \delta\Phi(\hat{\mathbf{x}}) + dx^a \mathcal{D}_a \Phi \end{aligned}$$

where $\delta\Phi(\hat{\mathbf{x}}') = \Phi'(\hat{\mathbf{x}}') - \Phi(\hat{\mathbf{x}}') \approx \Phi'(\hat{\mathbf{x}}) - \Phi(\hat{\mathbf{x}}) = \delta\Phi(\hat{\mathbf{x}})$ at the first order in ε , and $d\Phi = dx^a \mathcal{D}_a \Phi$ according to Eq. (3.9) of the 5D differential calculus. Since for scalar fields $\delta_T \Phi(\hat{\mathbf{x}}) = 0$, we get the relation

$$\Phi'(\hat{\mathbf{x}}) = \Phi(\hat{\mathbf{x}}) - dx^a \mathcal{D}_a \Phi \quad (\text{C.2})$$

Let us consider the *formal* Action for free scalar particles in κ -Minkowski spacetime at the finite 4-volume Ω

$$S[\Phi] = \frac{1}{2} \int_{\Omega} \left(\tilde{\mathcal{D}}_a \Phi \cdot \mathcal{D}^a \Phi - m^2 \Phi^2 \right). \quad (\text{C.3})$$

The translation-invariance of the action implies that the total variation of the action under the transformation Eq. (C.1)

$$\delta S_{\Omega}[\Phi] = \frac{1}{2} \int_{\Omega'} (\tilde{\mathcal{D}}_a \Phi' \cdot \mathcal{D}^a \Phi' - m^2 \Phi'^2)(\hat{\mathbf{x}}) - \frac{1}{2} \int_{\Omega} (\tilde{\mathcal{D}}_a \Phi \cdot \mathcal{D}^a \Phi - m^2 \Phi^2)(\hat{\mathbf{x}})$$

be zero. Because of the translation-invariance of the integral, we can write

$$\begin{aligned} \delta S_{\Omega}[\Phi] &= \frac{1}{2} \int_{\Omega} (\tilde{\mathcal{D}}_a \Phi' \cdot \mathcal{D}^a \Phi' - m^2 \Phi'^2)(\hat{\mathbf{x}} - \epsilon) - \frac{1}{2} \int_{\Omega} (\tilde{\mathcal{D}}_a \Phi \cdot \mathcal{D}^a \Phi - m^2 \Phi^2)(\hat{\mathbf{x}}) \\ &= \frac{1}{2} \int_{\Omega} (\tilde{\mathcal{D}}_a \Phi' \cdot \mathcal{D}^a \Phi' - m^2 \Phi'^2)(\hat{\mathbf{x}}) - \frac{1}{2} \int_{\Omega} d(\tilde{\mathcal{D}}_a \Phi' \cdot \mathcal{D}^a \Phi' - m^2 \Phi'^2)(\hat{\mathbf{x}}) \\ &\quad - \frac{1}{2} \int_{\Omega} (\tilde{\mathcal{D}}_a \Phi \cdot \mathcal{D}^a \Phi - m^2 \Phi^2)(\hat{\mathbf{x}}), \end{aligned}$$

and we get

$$\delta S_{\Omega}[\Phi] = -\frac{1}{2} \int_{\Omega} \left\{ d(\tilde{\mathcal{D}}_a \Phi \cdot \mathcal{D}^a \Phi - m^2 \Phi^2) - 2dx^a \mathcal{D}_a \mathcal{L} \right\}.$$

Let us consider the solution of EoM ($\Phi = \Phi_0$) in the previous equation

$$\delta S_{\Omega}[\Phi_0] = -\frac{dx_b}{2} \int_{\Omega} \left\{ \mathcal{D}^b [\tilde{\mathcal{D}}^a \Phi \cdot D_a \Phi - m^2 \Phi \cdot e^{\lambda P_0} \Phi] - 2\mathcal{D}^b \mathcal{L} \right\}_{\Phi=\Phi_0}. \quad (\text{C.4})$$

where $a, b = 0, 1, 2, 3, 4$.

Let us consider first the term in dx_β ($b = \beta$) in Eq. (C.4). Using Eq. (3.10) we obtain

$$\begin{aligned}\mathcal{D}^\beta[\tilde{D}^a\Phi \cdot D_a\Phi - m^2\Phi \cdot e^{\lambda P_0}\Phi] &= \mathcal{D}^\beta\tilde{D}^a\Phi \cdot e^{\lambda P_0}D_a\Phi - m^2\mathcal{D}^\beta\Phi \cdot e^{2\lambda P_0}\Phi \\ &\quad + e^{-\lambda P_0\delta_{\beta 0}}\tilde{D}^a\Phi \cdot \mathcal{D}^\beta D_a\Phi - m^2e^{-\lambda P_0\delta_{\beta 0}} \cdot e^{\lambda P_0}\mathcal{D}^\beta\Phi \\ &\quad - i\delta_{\beta 0}\lambda\tilde{D}_j\tilde{D}^a\Phi \cdot D_jD_a\Phi + i\delta_{\beta 0}\lambda m^2\tilde{D}_j\Phi \cdot e^{\lambda P_0}D_j\Phi.\end{aligned}$$

Using the coproduct of D^a , Eq. (B.4), and the equality $D^aD_a\Phi_0 = -m^2e^{\lambda P_0}\Phi_0$, we get

$$\begin{aligned}\mathcal{D}^\beta\tilde{D}^a\Phi_0 \cdot e^{\lambda P_0}D_a\Phi_0 &= D_a[e^{-\lambda P_0}\mathcal{D}^\beta\Phi_0 \cdot D^a\Phi_0] + m^2e^{-\lambda P_0}\mathcal{D}^\beta\Phi_0 \cdot e^{\lambda P_0}\Phi_0 \\ e^{-\lambda P_0\delta_{\beta 0}}\tilde{D}^a\Phi_0 \cdot \mathcal{D}^\beta D_a\Phi_0 &= D_a[e^{-\lambda P_0\delta_{\beta 0}}\tilde{D}^a\Phi_0 \cdot \mathcal{D}^\beta\Phi_0] + m^2e^{-\lambda P_0\delta_{\beta 0}}\Phi_0 \cdot e^{\lambda P_0}\mathcal{D}^\beta\Phi_0 \\ \tilde{D}_j\tilde{D}^a\Phi_0 \cdot D_jD_a\Phi_0 &= D_a[\tilde{D}_j\tilde{D}^a\Phi_0 \cdot D_j\Phi_0] + m^2\tilde{D}_j\Phi_0 \cdot e^{\lambda P_0}D_j\Phi_0\end{aligned}$$

Thus,

$$\begin{aligned}\mathcal{D}^\beta[\tilde{D}^a\Phi_0 \cdot D_a\Phi_0 - m^2\Phi_0 \cdot e^{\lambda P_0}\Phi_0] &= D_a\left[e^{-\lambda P_0}\mathcal{D}^\beta\Phi_0 \cdot D^a\Phi_0 + e^{-\lambda P_0\delta_{\beta 0}}\tilde{D}^a\Phi_0 \cdot \mathcal{D}^\beta\Phi_0\right. \\ &\quad \left.- i\delta_{\beta 0}\lambda\tilde{D}_j\tilde{D}^a\Phi_0 \cdot D_j\Phi_0 + ig_{a0}\lambda m^2e^{-\lambda P_0}\mathcal{D}^\beta\Phi_0 \cdot e^{\lambda P_0}\Phi_0\right].\end{aligned}$$

Moreover, we can write $\mathcal{D}^\beta = \frac{(1+e^{-\lambda P_0})^{\delta_{\beta 0}}}{2}D^\beta + i\delta_{\beta 0}\frac{\lambda}{2}e^{\lambda P_0}\vec{P}^2$ so that

$$\mathcal{D}^\beta\mathcal{L}[\Phi_0] = \frac{(1+e^{-\lambda P_0})^{\delta_{\beta 0}}}{2}D^\beta\mathcal{L}[\Phi_0] + i\delta_{\beta 0}\frac{\lambda}{2}e^{\lambda P_0}\vec{P}^2\mathcal{L}[\Phi_0], \quad (\text{C.5})$$

where we can ignore the last term since it does not contribute to the conserved quantities which are defined up to divergence terms, as in the commutative case.

Using the same procedure for the term in dx_4 ($b = 4$) in Eq. (C.4), we obtain

$$\begin{aligned}\mathcal{D}^4[\tilde{D}^a\Phi_0 \cdot D_a\Phi_0 - m^2\Phi_0 \cdot e^{\lambda P_0}\Phi_0] &= D_a\left[e^{-\lambda P_0}\mathcal{D}^4\Phi_0 \cdot D^a\Phi_0 + e^{-\lambda P_0}\tilde{D}^a\Phi_0 \cdot \mathcal{D}^4\Phi_0\right. \\ &\quad \left.- \lambda\tilde{D}^\nu\tilde{D}^a\Phi_0 \cdot D_\nu\Phi_0 + ig_{a0}\lambda m^2e^{-\lambda P_0}\mathcal{D}^4\Phi_0 \cdot e^{\lambda P_0}\Phi_0\right]_{\Phi_0}.\end{aligned}$$

(let us remember that \mathcal{D}^4 acts as a constant over Φ_0 : $\mathcal{D}^4\Phi_0 = \frac{\lambda}{2}m^2\Phi_0$). Moreover, we can write

$$\mathcal{D}^4\mathcal{L}[\Phi_0] = -\frac{\lambda}{2}\tilde{D}_0D_0\mathcal{L}[\Phi_0] + \frac{\lambda}{2}e^{\lambda P_0}\vec{P}^2\mathcal{L}[\Phi_0],$$

where we can ignore the last term as in Eq. (C.5).

The variation of the action (C.4) can be finally written as

$$\begin{aligned}\delta S &= -\frac{dx_\beta}{2}\int_\Omega D^\mu\left[e^{-\lambda P_0}\mathcal{D}^\beta\Phi \cdot D_\mu\Phi + e^{-\lambda P_0\delta_{\beta 0}}\tilde{D}_\mu\Phi \cdot \mathcal{D}^\beta\Phi - i\delta_{\beta 0}\lambda\tilde{D}_j\tilde{D}_\mu\Phi \cdot D_j\Phi\right. \\ &\quad \left.+ ig_{\mu 0}\lambda m^2e^{-\lambda P_0}\mathcal{D}^\beta\Phi \cdot e^{\lambda P_0}\Phi - g_{\mu\beta}(1+e^{-\lambda P_0})^{\delta_{\beta 0}}\mathcal{L}\right]_{\Phi=\Phi_0} \\ &\quad -\frac{dx_4}{2}\int_\Omega D^\mu\left[e^{-\lambda P_0}\mathcal{D}^4\Phi \cdot D_\mu\Phi + e^{-\lambda P_0}\tilde{D}_\mu\Phi \cdot \mathcal{D}^4\Phi - \lambda\tilde{D}_\mu\tilde{D}^\nu\Phi \cdot D_\nu\Phi\right. \\ &\quad \left.+ ig_{\mu 0}\lambda m^2e^{-\lambda P_0}\mathcal{D}^4\Phi \cdot e^{\lambda P_0}\Phi + g_{\mu 0}\lambda\tilde{D}_0\mathcal{L}\right]_{\Phi=\Phi_0}\end{aligned}$$

The continuity equation is $D^\mu J_{\mu b} = 0$ where

$$\begin{aligned} J_{\mu\beta} &= e^{-\lambda P_0} \mathcal{D}_\beta \Phi_0 \cdot D_\mu \Phi_0 + e^{-\lambda P_0 \delta_{\beta 0}} \tilde{D}_\mu \Phi_0 \cdot \mathcal{D}_\beta \Phi_0 - i \delta_{\beta 0} \lambda \tilde{D}_j \tilde{D}_\mu \Phi_0 \cdot D_j \Phi_0 + \\ &\quad + i g_{\mu 0} \lambda m^2 e^{-\lambda P_0} \mathcal{D}_\beta \Phi_0 \cdot e^{\lambda P_0} \Phi_0 - g_{\mu\beta} (1 + e^{-\lambda P_0})^{\delta_{\beta 0}} \mathcal{L}_{\Phi_0} \\ J_{\mu 4} &= e^{-\lambda P_0} \mathcal{D}_4 \Phi_0 \cdot D_\mu \Phi_0 + e^{-\lambda P_0} \tilde{D}_\mu \Phi_0 \cdot \mathcal{D}_4 \Phi_0 - \lambda \tilde{D}_\mu \tilde{D}^\nu \Phi_0 \cdot D_\nu \Phi_0 + \\ &\quad + i g_{\mu 0} \lambda m^2 e^{-\lambda P_0} \mathcal{D}_4 \Phi_0 \cdot e^{\lambda P_0} \Phi_0 + g_{\mu 0} \lambda \tilde{D}_0 \mathcal{L}_{\Phi_0} \end{aligned}$$

and the conserved currents are

$$\begin{aligned} J_{00} &= e^{-\lambda P_0} \mathcal{D}_0 \Phi_0 \cdot D_0 \Phi_0 + e^{-\lambda P_0} \tilde{D}_0 \Phi_0 \cdot \mathcal{D}_0 \Phi_0 - i \lambda \tilde{D}_j \tilde{D}_0 \Phi_0 \cdot D_j \Phi_0 \\ &\quad + i \lambda m^2 e^{-\lambda P_0} \mathcal{D}_0 \Phi_0 \cdot e^{\lambda P_0} \Phi_0 - (1 + e^{-\lambda P_0}) \mathcal{L}_{\Phi_0} \\ J_{0j} &= \tilde{D}_j \Phi_0 \cdot D_0 \Phi_0 + \tilde{D}_0 \Phi_0 \cdot D_j \Phi_0 + i \lambda m^2 \tilde{D}_j \Phi_0 \cdot e^{\lambda P_0} \Phi_0 \\ J_{04} &= e^{-\lambda P_0} \mathcal{D}_4 \Phi_0 \cdot D_0 \Phi_0 + e^{-\lambda P_0} \tilde{D}_0 \Phi_0 \cdot \mathcal{D}_4 \Phi_0 - \lambda \tilde{D}_0 \tilde{D}^\nu \Phi_0 \cdot D_\nu \Phi_0 \\ &\quad + i \lambda m^2 e^{-\lambda P_0} \mathcal{D}_4 \Phi_0 \cdot e^{\lambda P_0} \Phi_0 + i (1 - e^{-\lambda P_0}) \mathcal{L}_{\Phi_0} \end{aligned}$$

After some calculations they read as

$$\begin{aligned} J_{00} &= \tilde{D}_0 \Phi_0 \cdot D_0 \Phi_0 + \tilde{D}_j \Phi_0 \cdot D_j \Phi_0 + m^2 e^{-\lambda P_0} \Phi_0 \cdot e^{\lambda P_0} \Phi_0 - i J_{04} \\ J_{0j} &= \tilde{D}_j \Phi_0 \cdot D_0 \Phi_0 + \tilde{D}_0 \Phi_0 \cdot D_j \Phi_0 + i \lambda m^2 \tilde{D}_j \Phi_0 \cdot e^{\lambda P_0} \Phi_0 \\ J_{04} &= \frac{\lambda m^2}{2} (\tilde{D}_0 + i \lambda m^2) (e^{-\lambda P_0} \Phi_0 \cdot e^{\lambda P_0} \Phi_0) - \lambda \tilde{D}_0 \tilde{D}^\nu \Phi_0 \cdot D_\nu \Phi_0 + \lambda \tilde{D}_0 \mathcal{L}_{\Phi_0} \end{aligned}$$

To simplify the calculation of the conserved charges $Q_b = \int d^3x J_{0b}$ we introduce the following linear combinations of J_{0b} :

$$\begin{aligned} K_{00} &= J_{00} + i J_{04} = \tilde{D}_0 \Phi_0 \cdot D_0 \Phi_0 + \tilde{D}_j \Phi_0 \cdot D_j \Phi_0 + m^2 e^{-\lambda P_0} \Phi_0 \cdot e^{\lambda P_0} \Phi_0 \\ K_{0j} &= J_{0j} = \tilde{D}_j \Phi_0 \cdot D_0 \Phi_0 + \tilde{D}_0 \Phi_0 \cdot D_j \Phi_0 + i \lambda m^2 \tilde{D}_j \Phi_0 \cdot e^{\lambda P_0} \Phi_0 \\ K_{04} &= J_{04} = \frac{\lambda m^2}{2} (\tilde{D}_0 + i \lambda m^2) (e^{-\lambda P_0} \Phi_0 \cdot e^{\lambda P_0} \Phi_0) - \lambda \tilde{D}_0 \tilde{D}^\nu \Phi_0 \cdot D_\nu \Phi_0 + \lambda \tilde{D}_0 \mathcal{L}_{\Phi_0} \end{aligned} \quad (\text{C.6})$$

and in the next section we compute the conserved quantities $\tilde{Q}_b = \int d^3x K_{0b}$. We will then obtain Q_b by linear combinations of \tilde{Q}_b

$$Q_b = \tilde{Q}_b - i \delta_{b0} \tilde{Q}_4$$

D Construction of the Conserved Charges

By using the expression (4.26) for the field $\Phi(\hat{\mathbf{x}})$, we can write the conserved quantities \tilde{Q}_b associated with the currents K_{0b}

$$\tilde{Q}_b(\hat{\mathbf{x}}_0) = \int K_{0b}(\hat{\mathbf{x}}) = \frac{1}{2} \int d^3x \int d^4k d^4p \tilde{\phi}(k) K_{0b}(k_\mu, p_\mu) \tilde{\phi}(p) \delta[C_\lambda(k)] \delta[C_\lambda(p)] \hat{e}_k \hat{e}_p$$

where $K_{0b}(k_\mu, p_\mu)$ represent the Fourier-transforms of $K_{0b}(\hat{\mathbf{x}})$ (C.6).

The conserved quantities associated with the currents K_{0b} are

$$\tilde{Q}_b(\hat{\mathbf{x}}_0) = \int K_{0b}(\hat{\mathbf{x}}) = \frac{1}{2} \int d^3x \int d^4k d^4p \tilde{\phi}(k) K_{0b}(k_\mu, p_\mu) \tilde{\phi}(p) \delta[C_\lambda(k)] \delta[C_\lambda(p)] \hat{e}_k \hat{e}_p$$

where

$$\begin{aligned} K_{00}(k_\mu, p_\mu) &= \tilde{D}_0(k) D_0(p) + \tilde{D}_j(k) D_j(p) + m^2 e^{-\lambda k_0} e^{\lambda p_0} \\ K_{0j}(k_\mu, p_\mu) &= \tilde{D}_j(k) D_0(p) + \tilde{D}_0(k) D_j(p) + i\lambda m^2 \tilde{D}_j(k) e^{\lambda p_0} \\ K_{04}(k_\mu, p_\mu) &= \frac{\lambda m^2}{2} e^{\lambda(p_0 - k_0)} (\tilde{D}_0(k_0 + p_0) + i\lambda m^2) - \lambda \tilde{D}_0 \tilde{D}^\nu \Phi \cdot D_\nu \Phi \\ &\quad + \frac{\lambda}{2} \tilde{D}_0(k_0 + p_0) (\tilde{D}^\nu(k) D_\nu(p) - m^2 e^{-\lambda p_0}). \end{aligned} \quad (\text{D.1})$$

Integrating in d^3x we obtain

$$\tilde{Q}_b(\hat{\mathbf{x}}_0) = \frac{1}{2} \int d^4k d^4p \tilde{\phi}(k) K_{0b}(k_\mu, p_\mu) \tilde{\phi}(p) \delta[C_\lambda(k)] \delta[C_\lambda(p)] \delta^{(3)}(k + e^{-\lambda k_0} p) e^{i(k_0 + p_0) \hat{\mathbf{x}}_0}$$

and computing the $\delta^{(3)}k$ we get

$$\begin{aligned} \tilde{Q}_b(\hat{\mathbf{x}}_0) &= \frac{1}{2} \int dk_0 d^4p \tilde{\phi}(k_0, -e^{-\lambda k_0} p) K_{0b}(k_0, -e^{-\lambda k_0} p, p_0, p) \tilde{\phi}(p) e^{i(k_0 + p_0) \hat{\mathbf{x}}_0} \cdot \\ &\quad \delta \left(\frac{4}{\lambda^2} \sinh^2 \left(\frac{\lambda k_0}{2} \right) - e^{-\lambda k_0} \vec{p}^2 - m^2 \right) \delta \left(\frac{4}{\lambda^2} \sinh^2 \left(\frac{\lambda p_0}{2} \right) - e^{\lambda p_0} \vec{p}^2 - m^2 \right) \end{aligned}$$

By considering the product of the two delta functions, it is easy to see that only two solutions are possible $k_0^{(1)} = -p_0$ and $k_0^{(2)} = \lambda^{-1} \ln(2 + \lambda^2 m^2 - e^{-\lambda p_0})$. However, we can prove that the second solution does not contribute to the integral, in fact

$$\begin{aligned} K_{00}(k_0^{(2)}, -e^{-\lambda k_0^{(2)}} p, p_0, p) &= e^{-\lambda k_0} \left[-\frac{2}{\lambda^2} (\cosh(\lambda p_0) - 1) + e^{\lambda p_0} p_j^2 + m^2 \right] \\ &= -e^{-\lambda k_0} C_\lambda(p) = 0 \\ K_{0j}(k_0^{(2)}, -e^{-\lambda k_0^{(2)}} p, p_0, p) &= 0 \\ K_{04}(k_0^{(2)}, -e^{-\lambda k_0^{(2)}} p, p_0, p) &= 0 \end{aligned}$$

Thus, for $k_0 = k_0^{(2)}$, the integral is zero and we are left with the solution $k_0 = k_0^{(1)} = -p_0$. In this way the term dependent on $\hat{\mathbf{x}}_0$ disappears and \tilde{Q}_b turns out to be time-independent. Thus, \tilde{Q}_b take the form

$$\begin{aligned} \tilde{Q}_b &= \frac{1}{2} \int dk_0 d^4p \tilde{\phi}(\dot{-}p) K_{0b}(\dot{-}p_\mu, p_\mu) \tilde{\phi}(p) \delta \left(\frac{4}{\lambda^2} \sinh^2 \left(\frac{\lambda k_0}{2} \right) - e^{\lambda k_0} \vec{p}^2 - m^2 \right) \cdot \\ &\quad \cdot \delta \left(\frac{4}{\lambda^2} \sinh^2 \left(\frac{\lambda p_0}{2} \right) - e^{\lambda p_0} \vec{p}^2 - m^2 \right) \end{aligned}$$

and after some calculations they become

$$\tilde{Q}_b = \frac{1}{2} \int dk_0 d^4 p \tilde{\phi}(\dot{-}p) K_{0b}(\dot{-}p_\mu, p_\mu) \tilde{\phi}(p) \frac{\delta(k_0 + p_0)}{\lambda \left| \frac{4}{\lambda^2} e^{-\frac{\lambda p_0}{2}} \sinh\left(\frac{\lambda p_0}{2}\right) + m^2 \right|} \cdot \delta[C_\lambda(p)]$$

One can verify that for $k_0 = -p_0$ the functions $K_{0b}(\dot{-}p_\mu, p_\mu)$ are

$$\begin{aligned} K_{0\mu}(\dot{-}p_\mu, p_\mu) &= -i\lambda e^{\lambda p_0} D_\mu(p) \left(\frac{4}{\lambda^2} e^{-\frac{\lambda p_0}{2}} \sinh\left(\frac{\lambda p_0}{2}\right) + m^2 \right) \\ K_{04}(\dot{-}p_\mu, p_\mu) &= i \frac{\lambda^2 m^2}{2} e^{2\lambda p_0} \left(\frac{4}{\lambda^2} e^{-\frac{\lambda p_0}{2}} \sinh\left(\frac{\lambda p_0}{2}\right) + m^2 \right) \end{aligned}$$

Substituting this expression in \tilde{Q}_b we get

$$\begin{aligned} \tilde{Q}_\mu &= -\frac{i}{2} \int d^4 p \tilde{\phi}(\dot{-}p) \tilde{\phi}(p) e^{\lambda p_0} D_\mu(p) \operatorname{sgn}\left(\frac{2}{\lambda^2}(1 - e^{-\lambda p_0}) + m^2\right) \delta[C_\lambda(p)] \\ \tilde{Q}_4 &= \frac{i}{2} \int d^4 p \tilde{\phi}(\dot{-}p) \tilde{\phi}(p) e^{2\lambda p_0} \frac{\lambda m^2}{2} \operatorname{sgn}\left(\frac{2}{\lambda^2}(1 - e^{-\lambda p_0}) + m^2\right) \delta[C_\lambda(p)] \end{aligned}$$

where $\dot{-}p_\mu = (-p_0, -e^{\lambda p_0} p)$ is the antipode of p_μ and $\operatorname{sgn}(y) = y/|y|$ is the sign function.

The five conserved charges Q_b ($b = 0, 1, 2, 3, 4$.) can be obtained by linear combinations of \tilde{Q}_b :

$$Q_b = \tilde{Q}_b - i\delta_{b0}\tilde{Q}_4 = -\frac{i}{2} \int d^4 p \tilde{\phi}(\dot{-}p) \tilde{\phi}(p) e^{2\lambda p_0} \tilde{\mathcal{D}}_b(p) \operatorname{sgn}\left(\frac{2}{\lambda^2}(1 - e^{-\lambda p_0}) + m^2\right) \delta[C_\lambda(p)] \quad (\text{D.2})$$

where $\tilde{\mathcal{D}}_b(p)$ is defined as in the Eq. (B.3).

E κ -Minkowski Plane Waves

Let us consider the function $\tilde{\phi}(k)$ in Eq. (4.26) given by

$$\tilde{\phi}_p(k) = N_p \delta^{(3)}(p - k) H\left[k_0 - \ln\left(1 + \frac{\lambda^2 m^2}{2}\right)\right], \quad (\text{E.3})$$

where $H(x)$ is the Heaviside function (equal to 0 if $x \leq 0$ and equal to 1 if $x > 0$), and N_p is a suitable normalization which can be chosen in different ways as in the commutative case. To be simple, let us set $N_p = 1$.

In this appendix we want to show that the field $\Phi_0^{(p)}(\hat{\mathbf{x}})$

$$\Phi_0^{(p)}(\hat{\mathbf{x}}) = \int d^4 k \tilde{\phi}_p(k) e^{-ik\hat{\mathbf{x}}} e^{ik_0\hat{\mathbf{x}}_0} \delta[C_\lambda(p)] \quad (\text{E.4})$$

corresponds to a on-shell *plane wave* in κ -Minkowski. By substituting the function (E.3) in Eq. (E.4) we obtain

$$\begin{aligned} \Phi_0^{(p)}(\hat{\mathbf{x}}) &= \int d^4 k \delta^{(3)}(p - k) H\left[k_0 - \ln\left(1 + \frac{\lambda^2 m^2}{2}\right)\right] \delta\left(\frac{4}{\lambda^2} \sinh^2\left(\frac{\lambda k_0}{2}\right) - e^{\lambda k_0} k^2 - m^2\right) e^{-ik\hat{\mathbf{x}}} e^{ik_0\hat{\mathbf{x}}_0} \\ &= \frac{\lambda}{2} \int dk_0 H\left[k_0 - \ln\left(1 + \frac{\lambda^2 m^2}{2}\right)\right] \left(\frac{\delta(k_0 - w_+(p))}{\left|1 + \frac{\lambda^2 m^2}{2} - e^{-\lambda w_+}\right|} + \frac{\delta(k_0 - w_-(p))}{\left|1 + \frac{\lambda^2 m^2}{2} - e^{-\lambda w_-}\right|} \right) e^{-ip\hat{\mathbf{x}}} e^{ik_0\hat{\mathbf{x}}_0} \end{aligned}$$

where $w_{\pm}(p) = \lambda^{-1} \ln \left(\frac{1 + \frac{\lambda^2 m^2}{2} \pm \sqrt{\frac{\lambda^4 m^4}{4} + \lambda^2 (\bar{p}^2 + m^2)}}{1 - \lambda^2 \bar{p}^2} \right)$. Since $w_-(p) < \ln \left(1 + \frac{\lambda^2 m^2}{2} \right)$ then

$$\Phi_0^{(p)}(\hat{\mathbf{x}}) = \frac{\lambda}{2} \int dk_0 \frac{\delta(k_0 - w_+(p))}{|1 + \frac{\lambda^2 m^2}{2} - e^{-\lambda w_+(p)}|} e^{-ip\hat{\mathbf{x}}} e^{ik_0 \hat{\mathbf{x}}_0} = \frac{\lambda}{2} \frac{e^{-ip\hat{\mathbf{x}}} e^{iw_+(p)\hat{\mathbf{x}}_0}}{|1 + \frac{\lambda^2 m^2}{2} - e^{-\lambda w_+(p)}|}$$

and we see that the field $\Phi_0^{(p)}$ corresponds to a on-shell *plane wave* in κ -Minkowski.

The real version of the field $\Phi_0^{(p)}(\hat{\mathbf{x}})$ is

$$\Phi_0^{(p,real)}(\hat{\mathbf{x}}) = \frac{\lambda}{2} \frac{(e^{-ip\hat{\mathbf{x}}} e^{iw_+\hat{\mathbf{x}}_0} + h.c.)}{|1 + \frac{\lambda^2 m^2}{2} - e^{-\lambda w_+(p)}|} = \frac{\lambda}{2} \frac{(e^{-ip\hat{\mathbf{x}}} e^{iw_+\hat{\mathbf{x}}_0} + e^{ipe^{\lambda w_+\hat{\mathbf{x}}} e^{-iw_+\hat{\mathbf{x}}_0}})}{|1 + \frac{\lambda^2 m^2}{2} - e^{-\lambda w_+(p)}|}$$

where we have used the property (2.3) $\hat{e}(k))^{\dagger} = \hat{e}(\dot{-}k)$.

It is easy to see that the Fourier transform corresponding to the real plane wave $\Phi_0^{(p,real)}(\hat{\mathbf{x}})$ is

$$\tilde{\phi}_p(k) = \delta^{(3)}(k - p) H \left[k_0 - \ln \left(1 + \frac{\lambda^2 m^2}{2} \right) \right] + \delta^{(3)}(k \dot{+} p) H \left[-k_0 - \ln \left(1 + \frac{\lambda^2 m^2}{2} \right) \right].$$

Under the transformation $k_{\mu} \rightarrow \dot{-}k_{\mu}$ we get

$$\tilde{\phi}_p(\dot{-}k) = e^{-3\lambda k_0} \left[\delta^{(3)}(k \dot{+} p) H(-k_0 - \ln(1 + \frac{\lambda^2 m^2}{2})) + \delta^{(3)}(k - p) H(k_0 - \ln(1 + \frac{\lambda^2 m^2}{2})) \right].$$

References

- [1] W. Pauli, *Letter of Heisenberg to Peierls (1930)*, Scientific Correspondence, Vol II, p.15, Ed. Karl von Meyenn, Springer-Verlag, 1985.
- [2] H. S. Snyder, *Quantized Space-Time*, Phys. Rev. **71** (1947) 38.
- [3] L. J. Garay, *Quantum gravity and minimum length*, Int. J. Mod. Phys. A **10** (1995) 145.
- [4] S. Doplicher, K. Fredenhagen and J. E. Roberts, *The Quantum Structure of Space-Time at the Planck Scale and Quantum Fields*, Commun. Math. Phys. **172** (1995) 187;
- [5] S. Doplicher, K. Fredenhagen and J. E. Roberts, *Space-time quantization induced by classical gravity*, Phys. Lett. B **331** (1994) 39.
- [6] J. Lukierski, A. Nowicki, H. Ruegg and V. N. Tolstoy, *Q-deformation of Poincaré algebra*, Phys. Lett. B **264**, 331 (1991).
- [7] J. Lukierski, A. Nowicki and H. Ruegg, *New quantum Poincaré algebra and κ -deformed field theory*, Phys. Lett. B **293**, 344 (1992).
- [8] P. Kosinski and P. Maslanka, *The duality between κ -Poincaré algebra and κ -Poincaré group*, [hep-th/9411033].

- [9] P. Kosinski, J. Lukierski and P. Maslanka, *Local field theory on kappa-Minkowski space, star products and noncommutative translations*, Czech. J. Phys. **50** (2000) 1283.
- [10] M. Dimitrijevic, L. Jonke, L. Moller, E. Tsouchnika, J. Wess and M. Wohlgenannt, *Deformed field theory on kappa-spacetime*, Eur. Phys. J. C **31** (2003) 129;
- [11] S. Majid, *Hopf algebras for physics at the Planck scale*, Class. Quantum Grav. **5** (1988) 1587;
- [12] S. Majid, *Quantum Groups and Noncommutative Geometry*, J. Math. Phys. **41** (2000) 3892.
- [13] S. Majid and H. Ruegg, *Bicrossproduct structure of kappa Poincaré group and non-commutative geometry*, Phys. Lett. B **334** (1994) 348.
- [14] J. Lukierski, A. Nowicki and H. Ruegg, *Classical and quantum-mechanics of free κ -relativistic systems*, Ann. Phys. **243** (1995) 90.
- [15] A. Ballesteros, N. R. Bruno and F. J. Herranz, *A non-commutative Minkowskian spacetime from a quantum AdS algebra*, Phys. Lett. B **574** (2003) 276
- [16] A. Agostini, G. Amelino-Camelia and F. D'Andrea, *Hopf algebra description of noncommutative-spacetime symmetries*, Int. J. Mod. Phys. A **19** (2004) 5187-5219.
- [17] G. Amelino-Camelia, J. Lukierski and A. Nowicki, *kappa-deformed covariant phase space and quantum-gravity uncertainty relations*, Phys. Atom. Nucl. **61** (1998) 1811 [Yad. Fiz. **61** (1998) 1925];
- [18] G. Amelino-Camelia, J. Lukierski and A. Nowicki, *Distance measurement and kappa-deformed propagation of light and heavy probes*, Int. J. Mod. Phys. A **14** (1999) 4575.
- [19] G. Amelino-Camelia, *Relativity in space-times with short-distance structure governed by an observer-independent (Planckian) length scale*, Int. J. Mod. Phys. D **11** (2002) 35;
- [20] G. Amelino-Camelia, *Doubly special relativity*, Nature **418** (2002) 34.
- [21] A. Agostini, G. Amelino-Camelia, M. Arzano, A. Marciano and R. A. Tacchi, *Generalizing the Noether theorem for Hopf-algebra spacetime symmetries*, Mod. Phys. Lett. A **22** (2007) 1779
- [22] A. Sitarz, *Noncommutative differential calculus on the kappa Minkowski space*, Phys. Lett. B **349** (1995) 42
- [23] G. Amelino-Camelia, A. Marciano and D. Pranzetti, *On the 5D differential calculus and translation transformations in 4D kappa-Minkowski noncommutative spacetime*, arXiv:0709.2063 [hep-th].
- [24] A. Agostini, F. Lizzi and A. Zampini, *Generalized Weyl systems and kappa-Minkowski space*, Mod. Phys. Lett. A **17** (2002) 2105.
- [25] S. Meljanac, A. Samsarov, M. Stojic and K. S. Gupta, *Kappa-Minkowski space-time and the star product realizations*, Eur. Phys. J. C **53** (2008) 295
- [26] A. Agostini, *kappa-Minkowski representations on Hilbert spaces*, J. Math. Phys. **48** (2007) 052305

- [27] S. Majid, *Foundations of quantum group theory*, Cambridge University Press (2000).
- [28] J. Madore, J. Mourad and A. Sitarz, *Deformations of Differential Calculi*, Mod. Phys. Lett. A **12** (1997) 975;
- [29] A. Dimakis, J. Madore *Differential Calculi and Linear Connections*, arXiv:q-alg/9601023v1.
- [30] S. Zakrzewski, *Quantum Poincaré group related to the κ -Poincaré algebra*, J. Phys.A **27** (1994) 2075.
- [31] C. Gonera, P. Kosinski, P. Maslanka. *Differential calculi on quantum Minkowski space*, <http://arxiv.org/abs/q-alg/9602007>.
- [32] A. Agostini, *Fields and Symmetries in κ -Minkowski Noncommutative Spacetime*, arXiv:hep-th/0312305.
- [33] S. Majid, *On q -regularization*, Int. J. Math. Phys.A **5** (1990) 4689.
- [34] G. Amelino-Camelia and S. Majid, *Waves on noncommutative spacetime and gamma-ray bursts*, Int. J. Mod. Phys. A **15** (2000) 4301
- [35] H. Goldstein. *Classical Mechanics* Addison Wesley, Third edition.